

# $SL_2(\mathbb{Z})$ -TILING OF THE TORUS, COXETER-CONWAY FRIEZES AND FAREY TRIANGULATIONS

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**ABSTRACT.** The notion of  $SL_2$ -tiling is a generalization of that of classical Coxeter-Conway frieze pattern. We classify doubly antiperiodic  $SL_2$ -tilings that contain a rectangular domain of positive integers. Every such  $SL_2$ -tiling corresponds to a pair of frieze patterns and a unimodular  $2 \times 2$ -matrix with positive integer coefficients. We relate this notion to triangulated  $n$ -gons in the Farey graph.

## 1. INTRODUCTION

Frieze patterns were introduced and studied by Coxeter and Conway, [7, 6], in the 70's. A frieze pattern is an infinite array of numbers, bounding by two diagonals of 1's, such that every four adjacent numbers  $a, b, c, d$  forming a "small" square satisfy the relation  $ad - bc = 1$  called *the unimodular rule*; for an example see Figure 1. The *width* of the frieze is the number of diagonals between the bounding diagonals of 1's.

The fundamental Conway-Coxeter theorem [6] offers the following classification: *frieze patterns with positive integer entries of width  $n - 3$ , are in one-to-one correspondence with triangulations of a convex  $n$ -gon*; for a simple proof see [11]. More precisely, given a triangulated  $n$ -gon, one constructs a frieze of width  $n - 3$  as follows. The diagonal next to the diagonal of 1's is formed by the numbers of triangles incident at each vertex (taken cyclically).

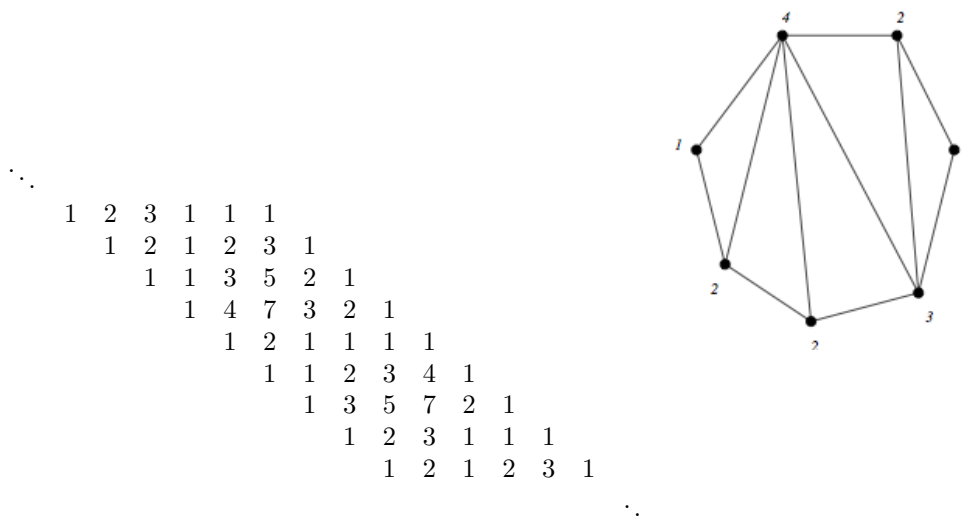


FIGURE 1. A 7-periodic frieze pattern and the corresponding triangulated heptagon.

*Key words and phrases.* Frieze pattern,  $SL_2$ -tiling, Farey graph, Modular group.

This, in particular, implies that every diagonal in a frieze of width  $n - 3$  is  $n$ -periodic. Throughout this paper, we will be considering frieze patterns with positive integer entries.

The following terminology is due to Conway and Coxeter [6]. A sequence of  $n$  positive integers  $q = (q_0, \dots, q_{n-1})$  is called a *quiddity* of order  $n$ , if there exists a triangulated  $n$ -gon such that every  $q_i$  is equal to the number of incident triangles at  $i$ -th vertex. For instance, the example in Figure 1 corresponds to the following quiddities of order 7:  $(1, 3, 2, 2, 1, 4, 2)$ ,  $(3, 2, 2, 1, 4, 2, 1), \dots$  (cyclic permutation).

Every quiddity of order  $n$  determines a unique positive integer frieze pattern. Two quiddities correspond to the same positive integer frieze pattern if and only if they differ by a cyclic permutation. According to the Conway-Coxeter theorem, positive integer frieze patterns can be enumerated by the Catalan numbers.

**Example 1.0.1.** For each case  $n = 3, 4$  and  $5$ , there is a unique (up to cyclic permutation) quiddity:  $(1, 1, 1)$ ,  $(1, 2, 1, 2)$  and  $(1, 3, 1, 2, 2)$ , respectively.

For  $n = 6$ , there are four different quiddities:

$$(1, 3, 1, 3, 1, 3), \quad (1, 4, 1, 2, 2, 2), \quad (1, 2, 3, 1, 2, 3), \quad (1, 3, 2, 1, 3, 2)$$

and their cyclic permutations.

We can also consider the “degenerate” case  $n = 2$ , where the corresponding “degenerate” quiddity is  $(0, 0)$ .

Examples of frieze patterns can be constructed using the computer program [17].

Among many beautiful properties of Coxeter-Conway friezes, the property of periodicity and so-called Laurent phenomenon are particularly important. They relate frieze patterns to the theory of cluster algebras developed by Fomin and Zelevinsky, [8, 9].

Various generalizations of Coxeter-Conway friezes have been recently introduced and studied, see [5, 16, 2, 1, 13]. One of the generalizations, called  $SL_2$ -tiling, was first considered by Bergeron and Reutenauer [3]. An  $SL_2$ -tiling is an infinite array of numbers satisfying the above unimodular rule, without the condition of bounding diagonals of 1's. Unlike the frieze patterns,  $SL_2$ -tilings are not necessarily periodic. Nevertheless, correspondences between  $SL_2$ -tilings and triangulations can be established, [12, 4].

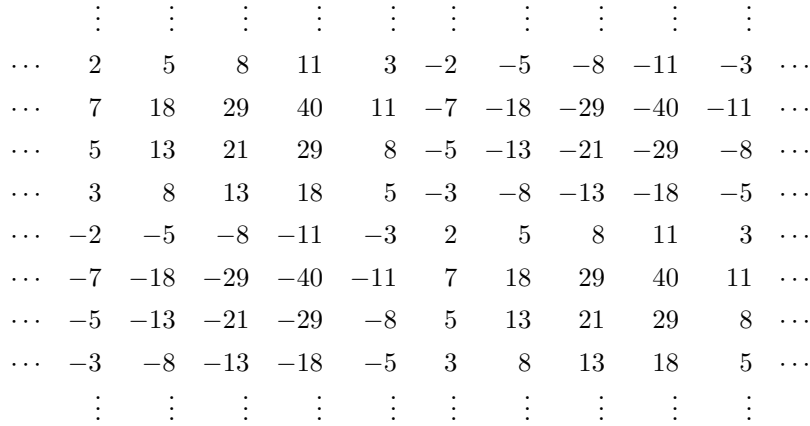


FIGURE 2. A  $(4, 5)$ -antiperiodic  $SL_2$ -tiling with positive rectangular domain.

The case of  $(n, m)$ -antiperiodic, or “toric”  $\mathrm{SL}_2$ -tilings was suggested in [3]. In this paper, we study such tilings.

The main results of the paper are the following.

We classify doubly antiperiodic  $\mathrm{SL}_2$ -tilings that contain a rectangular fundamental domain of positive integers. We show that every such  $\mathrm{SL}_2$ -tiling is generated by a pair of quiddities and a unimodular  $2 \times 2$ -matrix with positive integer coefficients. Although there are infinitely many such  $\mathrm{SL}_2$ -tilings, their description is very explicit.

Following the original idea of Coxeter [7], we also interpret the entries of a doubly periodic  $\mathrm{SL}_2$ -tiling that contain a rectangular fundamental domain of positive integers in terms of the Farey graph of rational numbers. Every such  $\mathrm{SL}_2$ -tiling corresponds to a pair: an  $n$ -gon and an  $m$ -gon in the Farey graph, and a totally positive matrix from  $\mathrm{SL}_2(\mathbb{Z})$  relating them. We also obtain an explicit formula for the entries of the tiling.

## 2. FAREY GRAPH AND THE CONWAY-COXETER THEOREM

In this section, we give an explanation of the relation between the Coxeter frieze patterns and triangulated  $n$ -gons.

It was already noticed by Coxeter [7] that a Farey series (of arbitrary order  $N$ ) defines a frieze pattern. Moreover, every frieze pattern corresponds to an  $n$ -gon (i.e., an  $n$ -cycle) in the Farey graph. A Farey  $n$ -gon always carries a triangulation; we will prove that this triangulation is precisely that of Conway-Coxeter theorem. This statement seems to be new and extend the observation illustrated in [17].

**2.1. Farey graph, Farey series and Farey  $n$ -gons.** For two rational numbers,  $r_1, r_2 \in \mathbb{Q}$ , written as irreducible fractions  $r_1 = \frac{a_1}{b_1}$  and  $r_2 = \frac{a_2}{b_2}$ , the *Farey distance* is defined by

$$d(r_1, r_2) := |a_1 b_2 - a_2 b_1|.$$

Recall the definition of the *Farey graph*.

- (1) The set of vertices of the Farey graph is  $\mathbb{Q} \cup \{\infty\}$ , with  $\infty$  represented by  $\frac{1}{0}$ .
- (2) Two vertices,  $r_1, r_2$  are joined by a (non-oriented) edge  $(r_1, r_2)$  whenever  $d(r_1, r_2) = 1$ .

The Farey graph is often embedded into the hyperbolic half-plane, the edges being realized as geodesics joining rational points on the ideal boundary.

The following classical properties of the Farey graph can be found in [10].

**Proposition 2.1.1.** *(i) Every 3-cycle of the Farey graph is of the form*

$$(2.1) \quad \left\{ \frac{a_1}{b_1}, \frac{a_1 + a_2}{b_1 + b_2}, \frac{a_2}{b_2} \right\}.$$

*(ii) Every edge of the Farey graph belongs to a 3-cycle.*

In other words, the Farey graph is triangulated.

The *Farey series* of order  $N$  is the sequence of irreducible fractions in  $[0, 1]$  whose denominators do not exceed  $N$ . We will write the sequences in the decreasing order; see Figure 3. Another classical result states that every Farey series is a cycle in the Farey graph.

**Proposition 2.1.2.** *Every two consecutive numbers in a Farey series are joined by an edge in the Farey graph.*

We will be interested in  $n$ -cycles in the Farey graph that are more general than Farey series.

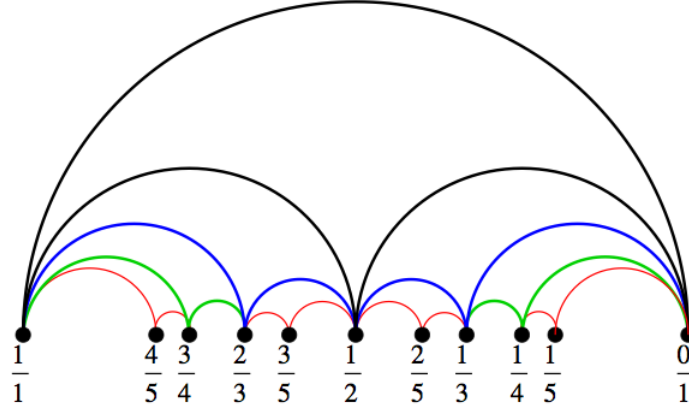


FIGURE 3. The Farey series of order 5 embedded in the Farey graph

**Definition 2.1.3.** (1) An  $n$ -gon in the Farey graph, or a *Farey  $n$ -gon* is a decreasing sequence of rationals  $(v_0, \dots, v_{n-1})$ :

$$\infty \geq v_0 > v_1 > \dots > v_{n-1} \geq 0,$$

such that every pair of consecutive numbers  $v_i, v_{i+1}$ , as well as  $v_{n-1}, v_0$ , are joined by an edge.

(2) The  $n$ -gon is called *normalized* if  $v_0 = \infty$  and  $v_{n-1} = 0$ .

Proposition 2.1.1 implies the following.

**Corollary 2.1.4.** *Every Farey  $n$ -gon is triangulated.*

We thus can speak of the *quiddity* of a Farey  $n$ -gon.

We define the notion of *cyclic equivalence* of Farey  $n$ -gons. Given an  $n$ -gon  $(v_0, \dots, v_{n-1})$ , consider the  $n$ -cycle  $(v_1, \dots, v_{n-1}, v_0)$ , and renormalize it using the  $\text{SL}_2(\mathbb{Z})$ -action so that  $v_1 = \infty$  and  $v_0 = 0$ . The obtained  $n$ -gon is called cyclically equivalent to the given one. For an example, see Figure 4.

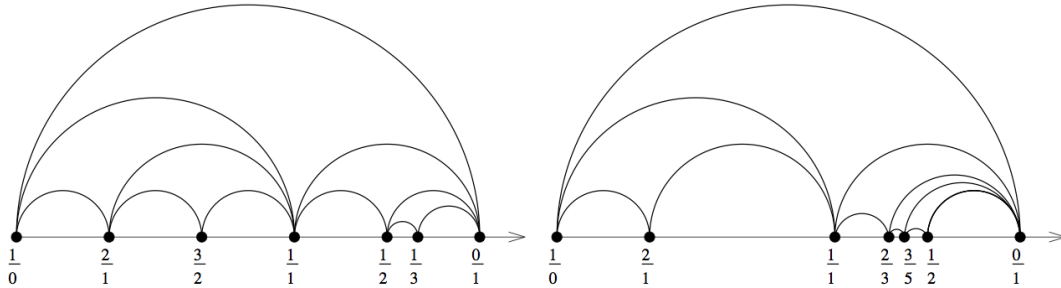


FIGURE 4. Two cyclically equivalent normalized heptagons in the Farey graph corresponding to the frieze of Figure 1

**2.2. Farey  $n$ -gons and Coxeter-Conway friezes.** Proposition 2.1.2 leads to the following observation due to Coxeter [7]: *every Farey series gives rise to a Coxeter-Conway frieze pattern of positive integers.* Along the same lines, we have the following strengthened statement.

**Proposition 2.2.1.** *The Coxeter-Conway frieze patterns of positive integers of width  $n - 3$  are in one-to-one correspondence with the normalized Farey  $n$ -gons, up to cyclic equivalence.*

*Proof.* The correspondence is given by considering the ratios of two consecutive rows of the frieze patterns. The sequence

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{a_1}{1}, \quad \dots, \quad v_i = \frac{a_i}{b_i}, \quad \dots, \quad v_{n-2} = \frac{1}{b_{n-2}}, \quad v_{n-1} = \frac{0}{1}$$

corresponds to the frieze determined by the rows

$$\begin{array}{ccccccc} 1 & a_1 & a_2 & \cdots & a_{n-3} & 1 & 0 \\ 0 & 1 & b_2 & & \cdots & b_{n-2} & 1 \end{array}$$

and *vice versa*. □

The Conway-Coxeter theorem mentioned in the introduction provides a relation between frieze patterns and triangulations. The following result somewhat “demystifies” this relation and provides an alternative proof of the Conway-Coxeter theorem.

**Theorem 1.** *The quiddity of a Farey  $n$ -gon coincides with the quiddity of the corresponding Coxeter-Conway frieze pattern.*

*Proof.* Consider a frieze pattern, and denote by  $c_{i,j}$  its entries:

$$\begin{array}{ccccccc} 0 & 1 & c_{1,1} & c_{1,2} & \cdots & c_{1,n-3} & 1 & 0 \\ & 0 & 1 & c_{2,2} & & \cdots & c_{2,n-2} & 1 \\ & & \ddots & \ddots & & & & \ddots \end{array}$$

where

$$\begin{cases} c_{i,j} = 1, & i - j = 1 \text{ or } 3 - n, \\ c_{i,j} = 0, & i - j = 2 \text{ or } 2 - n. \end{cases}$$

The quiddity of the frieze pattern reads in the  $n$ -periodic line  $(c_{i,i})$ .

Clearly, two consecutive rows determine the rest of the frieze; the following formula was proved in [7], formula (5.6):

$$c_{i,j} = c_{1,i-2}c_{2,j} - c_{1,j}c_{2,i-2}.$$

In particular, we have:

$$(2.2) \quad c_{i,i} = c_{1,i-2}c_{2,i} - c_{1,i}c_{2,i-2}.$$

The corresponding Farey  $n$ -gon has the following vertices

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{c_{1,1}}{1}, \quad \dots \quad v_i = \frac{c_{1,i}}{c_{2,i}}, \quad \dots \quad v_{n-2} = \frac{1}{c_{2,n-2}}, \quad v_{n-1} = \frac{0}{1}.$$

Therefore, the expression (2.2) reads:  $c_{i,i} = d(v_{i-2}, v_i)$ . It remains to calculate the Farey distance between pairs of vertices  $v_{i-2}$  and  $v_i$  in a Farey  $n$ -gon.

**Lemma 2.2.2.** *Given a (triangulated) Farey  $n$ -gon*

$$v_0 = \frac{1}{0}, \quad v_1 = \frac{a_1}{1}, \quad \dots, \quad v_i = \frac{a_i}{b_i}, \quad \dots, \quad v_{n-2} = \frac{1}{b_{n-2}}, \quad v_{n-1} = \frac{0}{1},$$

*the Farey distance  $d(v_{i-1}, v_{i+1})$  coincides with the number of triangles incident at  $v_i$ .*

*Proof.* Among all the vertices of the  $n$ -gon  $(v_i)$ , let us select those connected to  $v_i$  by edges of the Farey graph. Denote by  $\{v_{i_1}, \dots, v_{i_k}\}$ , resp.  $\{v_{i_{k+1}}, \dots, v_{i_{k+\ell}}\}$  the vertices at the left, resp. right, of  $v_i$ , so that

$$v_{i_1} > \dots > v_{i_k} > v_i > v_{i_{k+1}} > \dots > v_{i_{k+\ell}},$$

(note that  $v_{i_k} = v_{i-1}$  and  $v_{i_{k+1}} = v_{i+1}$ ). The number of triangles incident at  $v_i$  is then equal to  $k + \ell - 1$ .

Two consecutive selected vertices,  $v_{i_j}$  and  $v_{i_{j+1}}$  are connected by an edge. Indeed, this follows from the fact that every Farey polygon is triangulated. Therefore, the vertices  $(v_{i_j}, v_{i_{j+1}}, v_i)$  form a triangle (a 3-cycle) in the Farey graph. Using Eq. (2.1), we obtain by induction:

$$v_{i-1}(=v_{i_k}) = \frac{a_{i_1} + (k-1)a_i}{b_{i_1} + (k-1)b_i}, \quad v_{i+1}(=v_{i_{k+1}}) = \frac{a_{i_{k+\ell}} + (\ell-1)a_i}{b_{i_{k+\ell}} + (\ell-1)b_i}.$$

We have:

$$d(v_{i-1}, v_{i+1}) = a_{i_1}b_{i_{k+\ell}} - b_{i_1}a_{i_{k+\ell}} + (k-1)(a_i b_{i_{k+\ell}} - b_i a_{i_{k+\ell}}) + (\ell-1)(a_{i_1}b_i - b_{i_1}a_i).$$

By assumption,  $v_i$  is joined by edges with  $v_{i_1}$  and  $v_{i_{k+\ell}}$ , hence  $a_i b_{i_{k+\ell}} - b_i a_{i_{k+\ell}} = 1$ , and  $a_{i_1}b_i - b_{i_1}a_i = 1$ . Furthermore,  $(v_{i_1}, v_i, v_{i_{k+\ell}})$  is also a triangle, therefore  $a_{i_1}b_{i_{k+\ell}} - b_{i_1}a_{i_{k+\ell}} = 1$ . We have finally:

$$(2.3) \quad d(v_{i-1}, v_{i+1}) = k + \ell - 1.$$

Hence the lemma. □

Theorem 1 is proved. □

**2.3. Entries of the frieze pattern.** Coxeter's formula (5.6) in [7] for the entries of the frieze pattern translates into our language as the following general expression:

$$(2.4) \quad c_{i,j} = d(v_{i-2}, v_j),$$

where, as above,  $(v_i)$  is the Farey  $n$ -gon corresponding to the frieze pattern.

### 3. $SL_2$ -TILINGS

In this section, we introduce the main notions studied in this paper.

**3.1. Tame  $SL_2$ -tilings.** Let us first recall the notion of  $SL_2$ -tiling introduced in [3].

- (1) An  $SL_2$ -tiling, is an infinite matrix  $\mathcal{A} = (a_{i,j})_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}$ , such that every adjacent  $2 \times 2$ -minor equals 1:

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{vmatrix} = 1,$$

for all  $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ .

- (2) The tiling is called *tame* if every adjacent  $3 \times 3$ -minor equals 0:

$$\begin{vmatrix} a_{i,j} & a_{i,j+1} & a_{i,j+2} \\ a_{i+1,j} & a_{i+1,j+1} & a_{i+1,j+2} \\ a_{i+2,j} & a_{i+2,j+1} & a_{i+2,j+2} \end{vmatrix} = 0,$$

for all  $(i,j) \in \mathbb{Z} \times \mathbb{Z}$ .

Let us stress on the fact that a *generic*  $SL_2$ -tiling is tame.

**3.2. Antiperiodicity.** The following condition was also suggested in [3].

An  $SL_2$ -tiling is called  $(n, m)$ -antiperiodic if every row is  $n$ -antiperiodic, and every column is  $m$ -antiperiodic:

$$\begin{aligned} a_{i,j+n} &= -a_{i,j}, \\ a_{i+m,j} &= -a_{i,j}, \end{aligned}$$

for all  $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ .

The following relation between  $(n, m)$ -antiperiodic  $SL_2$ -tilings and the classical Coxeter-Conway frieze patterns shows that the antiperiodicity condition for the  $SL_2$ -tilings is natural and interesting.

**3.3. Frieze patterns and  $(n, n)$ -antiperiodic  $SL_2$ -tilings.** As explained in [3], every Coxeter-Conway frieze pattern of width  $n - 3$  can be extended to a tame  $(n, n)$ -antiperiodic  $SL_2$ -tiling, in a unique way.

The construction is as follows. One adds two diagonals of 0's next to the diagonals of 1's, and then continues by antiperiodicity.

**Example 3.3.1.** The frieze pattern in Figure 1 corresponds to the following  $(7, 7)$ -antiperiodic tame  $SL_2$ -tiling.

$$\begin{array}{cccccccccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & 2 & 3 & 1 & 1 & 1 & 0 & -1 & -2 & -3 & -1 & -1 & \cdots \\ \cdots & 0 & 1 & 2 & 1 & 2 & 3 & 1 & 0 & -1 & -2 & -1 & -2 & \cdots \\ \cdots & -1 & 0 & 1 & 1 & 3 & 5 & 2 & 1 & 0 & -1 & -1 & -3 & \cdots \\ \cdots & -2 & -1 & 0 & 1 & 4 & 7 & 3 & 2 & 1 & 0 & -1 & -4 & \cdots \\ \cdots & -1 & -1 & -1 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & -1 & \cdots \\ \cdots & -2 & -3 & -4 & -1 & 0 & 1 & 1 & 2 & 3 & 4 & 1 & 0 & \cdots \\ \cdots & -3 & -5 & -7 & -2 & -1 & 0 & 1 & 3 & 5 & 7 & 2 & 1 & \cdots \\ \cdots & -1 & -2 & -3 & -1 & -1 & -1 & 0 & 1 & 2 & 3 & 1 & 1 & \cdots \\ \cdots & 0 & -1 & -2 & -1 & -2 & -1 & -1 & 0 & 1 & 2 & 1 & 2 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

For the details of the above construction and the “antiperiodic nature” of Conway-Coxeter’s friezes; see [3, 14].

**3.4. Positive rectangular domain.** In this paper, we are considering  $(n, m)$ -antiperiodic  $SL_2$ -tilings that contain an  $m \times n$ -rectangular domain of positive integers.

More precisely, we are interested in  $SL_2$ -tilings of the following form:

$$(3.1) \quad \begin{array}{c|c|c|c|c} & \vdots & \vdots & \vdots & \\ \hline \cdots & P & -P & P & \cdots \\ \hline \cdots & -P & P & -P & \cdots \\ \hline & \vdots & \vdots & \vdots & \end{array}$$

where  $P$  is an  $m \times n$ -matrix with entries in  $\mathbb{Z}_{>0}$ . An example of such an  $SL_2$ -tiling is presented in Figure 2.

The following property is important for us.

**Proposition 3.4.1.** *An  $(n, m)$ -antiperiodic  $\mathrm{SL}_2$ -tiling that contains a positive  $m \times n$ -rectangular domain is tame.*

*Proof.* This is a consequence of the Jacobi identity or Dodgson formula on determinants:

$$\begin{vmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{vmatrix} \begin{vmatrix} \circ & \circ & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \circ \end{vmatrix} = \begin{vmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \\ \circ & \circ & \circ \end{vmatrix} \begin{vmatrix} \circ & \circ & \circ \\ \circ & \bullet & \bullet \\ \circ & \bullet & \bullet \end{vmatrix} - \begin{vmatrix} \circ & \circ & \circ \\ \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \end{vmatrix} \begin{vmatrix} \circ & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \circ \end{vmatrix}$$

where the white dots represent deleted entries, and the black dots initial entries.

Since the values are non zero and the  $2 \times 2$ -minors all equal to 1, the above identity implies that all the  $3 \times 3$ -minors vanish.  $\square$

#### 4. THE MAIN THEOREM

In this section, we formulate our main result. The proof will be given in Section 6.

**4.1. Classification.** It turns out that every  $\mathrm{SL}_2$ -tiling corresponds to a pair of frieze patterns and a positive integer  $2 \times 2$ -matrix  $M$  satisfying some conditions.

**Theorem 2.** *The set of  $(n, m)$ -antiperiodic  $\mathrm{SL}_2$ -tilings containing a fundamental rectangular domain of positive integers is in a one-to-one correspondence with the set of triples  $(q, q', M)$ , where*

$$q = (q_0, \dots, q_{n-1}), \quad q' = (q'_0, \dots, q'_{m-1})$$

*are quiddities of order  $n$  and  $m$ , respectively, and where  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is a unimodular  $2 \times 2$ -matrix with positive integer coefficients, such that the inequalities*

$$(4.1) \quad q_0 < \frac{b}{a}, \quad q'_0 < \frac{c}{a}$$

*are satisfied.*

**Remark 4.1.1.** It is important to notice that inequalities (4.1) also imply

$$(4.2) \quad q_0 < \frac{d}{c}, \quad q'_0 < \frac{d}{b}.$$

Indeed, the unimodular condition  $ad - bc = 1$  and the assumption that  $a, b, c, d$  are positive integers imply that  $\frac{b}{a} < \frac{d}{c}$  and  $\frac{c}{a} < \frac{d}{b}$ .

**Corollary 4.1.2.** *For every pair of quiddities  $q, q'$ , there exist infinitely many  $(n, m)$ -antiperiodic  $\mathrm{SL}_2$ -tilings containing a fundamental rectangular domain of positive integers.*

*Proof.* Given arbitrary pair of quiddities  $q$  and  $q'$ , the matrices:

$$\begin{pmatrix} 1 & b \\ c & bc + 1 \end{pmatrix}$$

satisfy (4.1) for sufficiently large  $b, c$ .  $\square$



**4.2. The semigroup  $\mathcal{S}$ .** Consider the set of  $2 \times 2$ -matrices with positive integral entries satisfying the following conditions of positivity:

$$(4.3) \quad \mathcal{S} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid \begin{array}{l} 0 < a < b < d, \\ 0 < a < c < d \end{array} \right\}.$$

Note that the inequalities  $b < d$  and  $c < d$  are included for the sake of completeness. These inequalities actually follow from  $a < b$ ,  $a < c$  together with  $ad - bc = 1$  and the assumption that  $a, b, c, d$  are positive.

We have the following property.

**Proposition 4.2.1.** *The set  $\mathcal{S} \subset \mathrm{SL}_2(\mathbb{Z})$  is a semigroup, i.e., it is stable by multiplication.*

*Proof.* Straightforward. □

The semigroup  $\mathcal{S}$  naturally appears in our context. Indeed, if  $n, m \geq 3$ , then the inequalities (4.1) imply  $M \in \mathcal{S}$ . Moreover every quiddity  $q$  contains a unit entry, so that after a cyclic permutation of any quiddity one can obtain  $q_0 = 1$ . The inequalities (4.1) then coincide with the conditions (4.3).

**4.3. Examples.** Let us give two simple examples of  $\mathrm{SL}_2$ -tilings.

**Example 4.3.1.** There is a one-to-one correspondence between  $(3, 3)$ -antiperiodic  $\mathrm{SL}_2$ -tilings containing a fundamental domain of positive integers and elements of the semigroup  $\mathcal{S}$ . Indeed, the only quiddity of order 3 is  $q = (1, 1, 1)$ . To every matrix (4.3) there corresponds the following  $\mathrm{SL}_2$ -tiling:

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ \cdots & a & & b & & b-a & \cdots \\ & & & & & & \\ \cdots & c & & d & & d-c & \cdots \\ & & & & & & \\ \cdots & c-a & & d-b & & d-b-c+a & \cdots \\ & \vdots & & \vdots & & \vdots & \end{array}$$

It is a good exercise to check that the positivity condition  $d - b - c + a > 0$  follows from (4.3) together with  $ad - bc = 1$ .

**Example 4.3.2.** In the case  $n = 2$  or  $m = 2$ , the conditions (4.1) become trivial.

Consider also the simplest (degenerate) case of  $(2, 2)$ -antiperiodic  $\mathrm{SL}_2$ -tilings. A  $(2, 2)$ -antiperiodic  $\mathrm{SL}_2$ -tiling containing a fundamental domain of positive integers is of the form:

$$\begin{array}{ccccccc} & \vdots & \vdots & \vdots & \vdots & & \\ \cdots & a & b & -a & -b & \cdots & \\ & & & & & & \\ \cdots & c & d & -c & -d & \cdots & \\ & \vdots & \vdots & \vdots & \vdots & & \end{array}$$

where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is an arbitrary unimodular matrix with positive integer coefficients. Note that this case corresponds to the “degenerate quiddity” of order 2, namely  $q = (0, 0)$ .

## 5. FRIEZE PATTERNS AND LINEAR RECURRENCE EQUATIONS

We will recall here a remarkable and well-known property of Coxeter-Conway frieze patterns. It concerns a relation of frieze patterns and linear recurrence equations. The statement presented in this subsection was implicitly obtained in [6]; for details see [14]. We recall this statement without proof.

### 5.1. Discrete non-oscillating Hill equations.

**Definition 5.1.1.** Let  $(c_i)_{i \in \mathbb{Z}}$  be an arbitrary  $n$ -periodic sequence of numbers.

(a) A linear difference equation

$$(5.1) \quad V_{i+1} = c_i V_i - V_{i-1},$$

where the sequence  $(c_i)$  is given (the coefficients) and where  $(V_i)$  is unknown (the solution), is called a discrete Hill, or Sturm-Liouville, or one-dimensional Schrödinger equation.

(b) The equation (5.1) is called *non-oscillating* if every solution  $(V_i)$  is antiperiodic:

$$V_{i+n} = -V_i,$$

for all  $i$ , and has exactly one sign change in any sequence  $(V_i, V_{i+1}, \dots, V_{i+n})$ .

In other words, every solution of a non-oscillating equation must have non-negative intervals of length  $n$ , that is,  $n$  consecutive non-negative values:  $(V_k, \dots, V_{k+n-1})$ .

Moreover, for *generic* solution of (5.1), all the elements  $V_j$  of a non-negative interval are *strictly positive*. Zero values can only occur at the endpoints:  $V_k = 0$ , or  $V_{k+n-1} = 0$ .

Note also that the coefficients in a non-oscillating equation are necessarily positive.

**5.2. Frieze patterns and difference equations.** The relation between the equations (5.1) and Coxeter-Conway frieze patterns is as follows.

**Proposition 5.2.1.** *Given an equation (5.1) with integer coefficients, it is a non-oscillating equation if and only if the coefficients  $(c_0, c_1, \dots, c_{n-1})$  form a quiddity.*

*Proof.* This is an immediate consequence of properties established by Coxeter and Conway. Indeed, it was proved in [7] (see also [6] property (17)) that the entries in any row of the pattern (extended by antiperiodicity) form a solution of an equation (5.1), where the coefficients  $c_i$  are given by the sequence on the first non-trivial diagonal. Thus, from a non-oscillating equation one can write down a frieze, and vice versa.

$$\begin{array}{cccccccc}
 & \ddots & & \ddots & & \ddots & & \ddots \\
 & & 1 & c_0 & \cdots & 1 & 0 & -1 & \cdots \\
 & & & & 1 & c_1 & \cdots & 1 & 0 & -1 & \cdots \\
 & & & & & & 1 & c_2 & \cdots & 1 & 0 & -1 & \cdots \\
 & & & & & & & & \ddots & & \ddots & & \ddots
 \end{array}$$

Finally, the integer condition establish the correspondence with quiddities. □

Of course, for an arbitrary non-oscillating equation (5.1), the corresponding frieze pattern does not necessarily have integer entries. In [14], the space of frieze patterns and the space of non-oscillating equation (5.1) are identified in a more general setting.

**Example 5.2.2.** (a) The simplest quiddity  $q = (1, 1, 1)$  corresponds to the non-oscillating equation with all  $c_i = 1$ . Every solution of this equation is 3-antiperiodic and can be obtained as a linear combination of the following two solutions:

$$(V_i^{(1)}) = (\dots, 0, 1, 1, 0, -1, -1, \dots), \quad (V_i^{(2)}) = (\dots, 1, 1, 0, -1, -1, 0, \dots).$$

This corresponds to a degenerate frieze of Coxeter-Conway of width 0. (b) The frieze from Figure 1 corresponds to the non-oscillating equation with 7-antiperiodic solutions that are linear combinations of the following two:

$$(V_i^{(1)}) = (\dots, 1, 2, 3, 1, 1, 1, 0, \dots), \quad (V_i^{(2)}) = (\dots, 0, 1, 2, 1, 2, 3, 1, \dots).$$

The above two solutions are exactly the first two rows of the frieze in Figure 1. One can of course choose different rows for a basis.

Note that, in the both cases, the basis solutions  $(V_i^{(1)}), (V_i^{(2)})$  are not generic since they contain zeros.

## 6. PROOF OF THEOREM 2

**6.1. The construction.** Given a triple  $(q, q', M)$  as in Theorem 2, we will construct an  $SL_2$ -tiling satisfying the above conditions. Define  $T = (a_{i,j})$  using the following recurrence relations:

$$(6.1) \quad \begin{aligned} a_{i,j+1} &:= q_j a_{i,j} - a_{i,j-1}, \\ a_{i+1,j} &:= q'_i a_{i,j} - a_{i-1,j}, \end{aligned}$$

for all  $i, j \in \mathbb{Z}$ , where the quiddities are periodically extended, i.e.  $q_i = q_{i+n}, q'_i = q'_{i+m}$ , and taking the initial conditions

$$(6.2) \quad \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is very easy to check that the tiling  $T$  is well-defined, i.e., the two recurrences commute and the calculations along the rows and columns give the same result. We show that the defined tiling  $T$  contains a fundamental rectangular domain of positive integers.

By Proposition 5.2.1, the defined tiling  $T$  is  $(n, m)$ -antiperiodic. Consider the following  $m \times n$ -subarray of  $T$

$$(6.3) \quad P = \begin{pmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1} \end{pmatrix}.$$

The main step of the proof of Theorem 2 is the following lemma.

**Lemma 6.1.1.** *The entries of  $P$  are positive integers.*

*Proof.* It turns out that thanks to Proposition 5.2.1 we will only need to perform “local” calculation of the elements neighboring to the initial ones:

$$\begin{array}{c|cc} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ \hline a_{0,-1} & a & b \\ a_{1,-1} & c & d \end{array}$$

The conditions (4.1) imply:  $a_{0,-1} < 0$  and  $a_{-1,0} < 0$ . Indeed, from (6.1) and (6.2), one has

$$a_{0,-1} = q_0 a - b, \quad a_{-1,0} = q'_0 a - c.$$

Since the rows and the columns of  $P$  are solutions of non-oscillating equations, and  $a$  is positive, this implies that all the values of the first row and the first column of  $P$  are positive.

Furthermore, again from the recurrence (6.1), one has

$$a_{-1,-1} = q_0 q'_0 a - q_0 c - q'_0 b + d.$$

The condition (4.1) then implies  $a_{-1,-1} > 0$ . Indeed, one establishes

$$0 < q_0 = a q_0 (d - q'_0 b) - b q_0 (c - q'_0 a) < b(d - q'_0 b) - b q_0 (c - q'_0 a) = b(q_0 q'_0 a - q_0 c - q'_0 b + d).$$

Proposition 5.2.1 then guarantees that

$$\begin{aligned} a_{0,-1} < 0, \quad \dots, \quad a_{m-1,-1} < 0, \\ a_{-1,0} < 0, \quad \dots, \quad a_{-1,n-1} < 0, \end{aligned}$$

and applying again Proposition 5.2.1, we deduce that all the entries in  $P$  are positive.  $\square$

**6.2. From tilings to triples.** Conversely, consider a  $(n, m)$ -periodic  $\text{SL}_2$ -tiling  $T = (a_{i,j})_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}$  such that the  $m \times n$ -subarray  $P$  given by (6.3) consists in positive integers. We claim that  $T$  can be obtained by the above construction.

**Lemma 6.2.1.** *The ratios of the first two rows of  $P$  form a decreasing sequence:*

$$\frac{a_{0,0}}{a_{1,0}} > \frac{a_{0,1}}{a_{1,1}} > \dots > \frac{a_{0,n-1}}{a_{1,n-1}},$$

and similarly for the ratios of the first two columns of  $P$ :

$$\frac{a_{0,1}}{a_{0,0}} > \frac{a_{1,1}}{a_{1,0}} > \dots > \frac{a_{m-1,1}}{a_{m-1,0}}.$$

*Proof.* This follows from the unimodular conditions  $a_{0,j}a_{1,j+1} - a_{0,j+1}a_{1,j} = 1$  and the assumption that all the entries of  $P$  are positive.  $\square$

**Lemma 6.2.2.** *The entries of  $T$  satisfy the recurrence relations (6.1) where  $q = (q_j)$  and  $q' = (q'_i)$  are  $n$ -periodic and  $m$ -periodic sequences of positive integers, respectively.*

*Proof.* Given  $(i, j)$ , there is a linear relation

$$\begin{pmatrix} a_{i,j+1} \\ a_{i+1,j+1} \end{pmatrix} = \lambda_{i,j} \begin{pmatrix} a_{i,j} \\ a_{i+1,j} \end{pmatrix} + \mu_{i,j} \begin{pmatrix} a_{i,j-1} \\ a_{i+1,j-1} \end{pmatrix}.$$

Using the  $\text{SL}_2$  conditions one immediately obtains the values

$$\lambda_{i,j} = a_{i,j-1}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j-1}, \quad \mu_{i,j} = -1.$$

From Lemma 6.2.1, one has  $\lambda_{i,j} > 0$ . Furthermore, it readily follows from the tameness property (see Proposition 3.4.1) that  $\lambda_{i,j}$  actually does not depend on  $i$ , so we use the notation  $q_j := \lambda_{i,j}$ .

The arguments for the rows are similar.  $\square$

**Lemma 6.2.3.** *The above sequences  $(q_0, \dots, q_{m-1})$  and  $(q'_0, \dots, q'_{n-1})$  are quiddities.*

*Proof.* The rows, resp. columns, of  $T$  are antiperiodic solutions of an equation (5.1) with  $c_i = c_{i+n} = q_i$ , resp.  $c_i = c_{i+m} = q'_i$ . It follows from Proposition 5.2.1 that the coefficients are quiddities.  $\square$

**Lemma 6.2.4.** *The  $2 \times 2$  left upper block of  $P$ , satisfies*

$$\begin{aligned} q_0 a_{0,0} &< a_{0,1}, \\ q'_0 a_{0,0} &< a_{1,0}. \end{aligned}$$

*Proof.* By antiperiodicity,  $a_{0,-1} < 0$ . One has from (6.1):  $a_{0,1} = q_0 a_{0,0} - a_{0,-1}$ , and similarly for  $q'_0$ . Hence the result.  $\square$

In other words, the elements of the matrix

$$\begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfy (4.1).

Theorem 2 is proved.

## 7. $\mathrm{SL}_2$ -TILINGS AND THE FAREY GRAPH

In this section, we give an interpretation of the entries  $a_{i,j}$  of a doubly periodic  $\mathrm{SL}_2$ -tiling. We follow the idea of Coxeter [7] and consider  $n$ -gons in the classical Farey graph.

**7.1. The distance between two  $n$ -gons.** Consider a doubly periodic  $\mathrm{SL}_2$ -tiling  $T = (a_{i,j})$  and the corresponding triple  $(q, q', M)$  (see Theorem 2). Our next goal is to give an explicit expression for the numbers  $a_{i,j}$  similar to (2.4).

From the triple  $(q, q', M)$  we construct the unique  $n$ -gon  $(v_0, v_1, \dots, v_{n-1})$  and the unique  $m$ -gon  $(v'_0, v'_1, \dots, v'_{m-1})$  with the “initial” conditions:

$$(v_0, v_1) := \left(\frac{a}{c}, \frac{b}{d}\right), \quad (v'_0, v'_{m-1}) := \left(\frac{1}{0}, \frac{0}{1}\right),$$

and with the quiddities  $(q_0, \dots, q_{n-1})$  and  $(q'_1, \dots, q'_m)$ , respectively. Notice that the quiddity  $q'$  is shifted cyclically.

**Theorem 3.** *The entries of the  $\mathrm{SL}_2$ -tiling  $T = (a_{i,j})$  are given by*

$$a_{i,j} = d(v'_{i-1}, v_j),$$

for all  $0 \leq i \leq m-1$ ,  $0 \leq j \leq n-1$ .

*Proof.* The main idea of the proof is to include the  $n$ -gon  $v$  and the  $m$ -gon  $v'$  into a bigger  $N$ -gon in a Farey graph, and then apply Eq. (2.4). In other words, we will include the fundamental domain  $P$  into a (bigger) frieze pattern.

First, let us show that

$$v'_{m-2} > v_0 > v_1 > \dots > v_{n-1} > v'_{m-1}.$$

Indeed, the vertices  $v'_{m-2}, v'_{m-1}, v'_0$  are consecutive vertices of the  $m$ -gon  $v'$ . By assumption,  $v'_{m-1} = \frac{0}{1}$ , so that the condition

$$d(v'_{m-2}, v'_{m-1}) = 1$$

implies  $v'_{m-2} = \frac{1}{\ell}$  for some  $\ell$ . By Lemma 2.2.2, the distance  $d(v'_0, v'_{m-2})$  coincides with the number of triangles at the vertex  $v'_{m-1}$  which is, by construction, equal to  $q'_0$ . We finally have:

$$d(v'_0, v'_{m-2}) = \ell = q'_0,$$

so that  $v'_{m-2} = \frac{1}{q'_0}$ . The inequality  $v'_{m-1} > v_0$  then follows from the second inequality (4.1).

It is well-known that the Farey graph is connected; see [10]. Therefore, two disjoint polygons,  $v$  and  $v'$ , belong to some  $N$ -gon that contain the  $n$ -gon  $v$  and the  $m$ -gon  $v'$ .

Theorem 3 then follows from formula (2.4).  $\square$

**Example 7.1.1.** Consider the tiling given in Figure 2. It corresponds to the following data

$$q = (1, 2, 2, 1, 3), \quad q' = (2, 1, 2, 1), \quad M = \begin{pmatrix} 2 & 5 \\ 7 & 18 \end{pmatrix}.$$

The associated 5-gon and 4-gon in the Farey graph are as follows:

$$v = \left( \frac{2}{7}, \frac{5}{18}, \frac{8}{29}, \frac{11}{40}, \frac{3}{11} \right), \quad \text{and} \quad v' = \left( \frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{0}{1} \right),$$

respectively. They can be included in an 11-gon; see Figure 5.

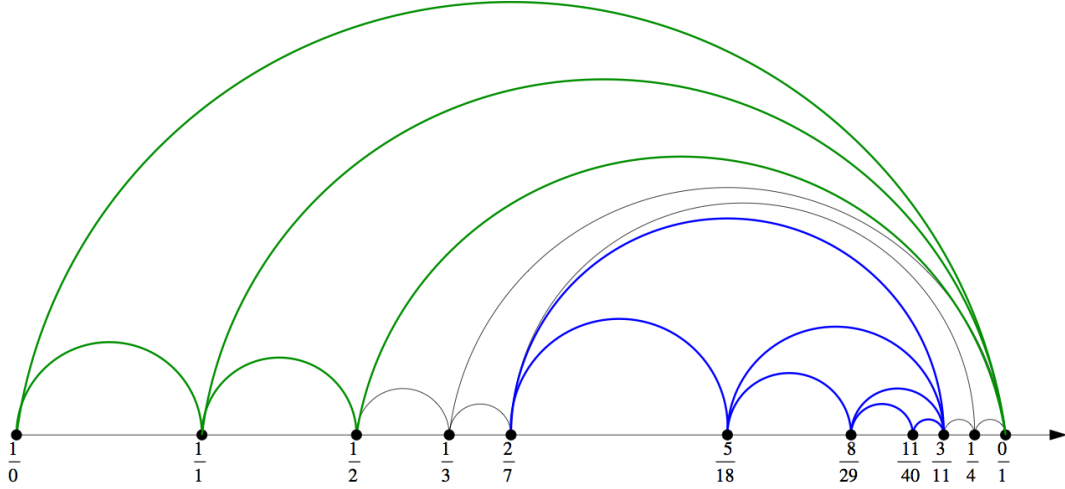


FIGURE 5. The subgraph associated with the tiling in Figure 2

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